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FAST TRACK COMMUNICATION

On the generation of anomalous diffusion

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Abstract
A superposition mechanism for the generation of anomalous diffusion, both subdiffusive and superdiffusive, is established. We consider a general system model in which a probe is tossed into a stochastic bath, and is constantly impacted by random gusts. All gusts affect the probe by a statistically common, yet arbitrary, impact pattern representing the generic gusts–probe interaction. Each gust has its own impact parameters—amplitude, frequency and initiation epoch. The probe's trajectory is the superposition of all gust impacts affecting it. We characterize the class of impact parameter statistics which produce anomalous diffusion probe trajectories for whatever impact patterns applied. This class of 'bath statistics' generates anomalous diffusion in a universal fashion—indifferent to the details of the gusts–probe interaction.

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For over a century diffusive motions have dominated the world of random transport processes. Diffusive motions are characterized by a linear temporal growth of their mean square displacements (MSDs). Namely if $\xi = (\xi(t))_{t\geqslant 0}$ is the trajectory of a diffusive motion, then $\langle \xi(t)^2\rangle \approx Dt$ where D is the motion's diffusion coefficient. Statistical models of diffusion include random walks, Brownian motion and finite-variance Lévy motions.

Recent decades have experienced growing scientific interest in so-called anomalous diffusion random transport mechanisms [1, 2]. Anomalous diffusions are characterized by a power-law temporal growth of their MSDs. Namely if $\xi = (\xi(t))_{t\geqslant 0}$ is the trajectory of an anomalous diffusion, then $\langle \xi(t)^2\rangle \approx Dt^\alpha$ where the coefficient D and the exponent α are positive parameters. Anomalous diffusion with exponent $0 < \alpha < 1$ disperses slower than diffusion, and is thus termed 'subdiffusion'. On the other hand, anomalous diffusion with exponent $\alpha > 1$ disperses faster than diffusion, and is thus termed 'superdiffusion'. In the case of superdiffusion the experimentally observed values of the exponent α are typically in the range $1 < \alpha < 2$.

Examples of subdiffusion include the transport of charge carriers in amorphous semiconductors [3], the propagation of contaminants in groundwater [4] and the movement of proteins in intracellular media [5, 6]. Essentially, there are two main statistical models yielding subdiffusion: Lévy halts, and anti-persistent fractional Brownian motion (fBm). In the first model random halts are introduced into the propagation of a 'standard' diffusive transport process. The halts occur randomly (at some given rate), and their durations are Lévy-distrusted—with fat-tailed probabilities and infinite means. These Lévy halts 'slow down' the transport from diffusive to subdiffusive [7, 8]. Perhaps the best known example of Lévy halts is the continuous time random walk (CTRW) [9] with Lévy waiting times. The second model is fBm with Hurst exponent H in the range $0 < H < 1/2$ [10]. In this Hurst range the increments of fBm are negatively correlated—causing its fluctuations to be anti-persistent, and the overall motion to display a subdiffusive statistical behavior (with exponent $\alpha = 2H$ taking values in the range $0 < \alpha < 1$).

Examples of superdiffusion include the transport of tracers in turbulent flows [11, 12], the search patterns of foraging animals [13] and particle trajectories in some cellular systems [14]. As in the case of subdiffusion, there are essentially two main statistical models yielding superdiffusion: Lévy walks, and persistent fBm. In the first model random ballistic walks are introduced into the propagation of a 'standard' diffusive transport process. The walks occur randomly (at some given rate), their velocities are random (usually +1 or −1 with equal probabilities) and their durations are Lévy-distrusted (with fat-tailed probabilities and infinite means). These Lévy walks 'speed up' the transport from diffusive to superdiffusive [15]. The second model is fBm with Hurst exponent H in the range $1/2 < H < 1$ [10]. In this Hurst range the increments of fBm are positively correlated—causing its fluctuations to be persistent, and the overall motion to display a superdiffusive statistical behavior (with exponent $\alpha = 2H$ taking values in the range $1 < \alpha < 2$).

Brownian motion—the quintessential example of diffusion—is a macroscopic manifestation of a microscopic phenomenon. Indeed, as observed by Sir Robert Brown, the jagged and erratic trajectory of a pollen particle suspended in liquid is caused by the 'bombardment effect' of trillions of molecules hitting the particle at random. As a general conceptual model of such a random motion, consider the trajectory of a probe tossed into a turbulent stochastic 'bath'. The probe is constantly impacted by random 'gusts'—these impacts generating the probe's random trajectory $Y = (Y(t))_{t \geqslant 0}$. A fairly general mathematical model for the random trajectory Y is the following impact superposition model:

$$Y(t) = \sum_{\tau_k \leqslant t} a_k X_k(\omega_k(t - \tau_k)), \tag{1}$$

where $X_k = (X_k(t))_{t \geqslant 0}$ is the random pattern by which gust k affects the probe, and (a_k, ω_k, τ_k) are the random 'impact parameters' of gust k—amplitude a_k (real valued), frequency ω_k (positive valued) and initiation epoch τ_k (non-negative valued). We henceforth assume that the random patterns $\{X_k\}$ are independent and identically distributed (i.i.d.) copies of a generic random 'impact pattern' $X = (X(t))_{t \geqslant 0}$, which describes the effect of a single arbitrary gust on the probe's trajectory.

In the special case the impact pattern X follows an exponential decay (i.e. $X(t) = \exp(-t)$)—or, more generally, when the impact pattern X is a deterministic function decaying to zero (as $t \to \infty$)—then the resulting random trajectory Y is a shot noise process [16]. The probe can also represent a single molecule embedded in solid and performing a spectral diffusion [17]—in which case the random trajectory Y represents the probe's fluctuating energy levels. In the context of signal processes, the superposition model of equation (1) can be interpreted as an aggregative communication model: the random trajectory Y representing the superimposed output signal of a communication channel 'fed' by the random input signals $\{X_k\}$—signal k transmitted with amplitude a_k, frequency ω_k and initiation epoch τ_k.

Ubiquitously observed physical phenomena—such as diffusion and anomalous diffusion—often stem from universal mechanisms, namely mechanisms that generate the phenomena under consideration in a 'universal fashion' which is indifferent to the particularities of the underlying details. For example, in the case of Brown's experiment, the observed Brownian motion is independent of the physical and chemical details of the pollen particles and the liquid used—any other particles (that can be suspended in liquid) and any other liquid would yield Brownian motion as well (albeit with a different diffusion coefficient D).

In the superposition model of equation (1) the gusts–probe interaction is represented by the impact pattern X. A universal behavior of the MSD of the probe's trajectory Y needs to be indifferent to the details of the impact pattern X. Hence, universality should stem from the statistics of the stochastic bath—conveyed by the statistics of the random impact parameters. In what follows we *seek statistics of the impact parameters that render the variance of the probe's trajectory Y independent of the choice of the impact pattern X*. To that end let us first describe the statistics of the impact parameters.

The set of impact parameters $\mathcal{P} = \{(a_k, \omega_k, \tau_k)\}_k$ forms a collection of points scattered arbitrarily on the three-dimensional domain $\mathcal{D} = (-\infty, \infty) \times (0, \infty) \times [0, \infty)$. The common statistical method for the random scattering of points in general domains is that of Poisson point processes [18], and we henceforth consider the impact parameters $\mathcal{P}$ to be a Poisson point process with intensity $\lambda(a, \omega, \tau)$. The Poissonian intensity $\lambda(a, \omega, \tau)$ governs the statistics of the impact parameters $\mathcal{P}$ [18]: (i) the number of gusts with impact parameters residing in a sub-domain D (of the domain $\mathcal{D}$) is Poisson distributed with mean $\iiint_D \lambda(a, \omega, \tau)\, \mathrm{d}a\, \mathrm{d}\omega\, \mathrm{d}\tau$; (ii) the number of gusts with impact parameters residing in disjoint sub-domains (of the domain $\mathcal{D}$) is independent random variables[4].

Poisson point processes have a wide spectrum of applications ranging from insurance and finance [19] to queueing systems [20]. In recent years we applied Poissonian statistics to study various aspects of stochastic fractality. Examples include fractality of random populations [21–23]; statistical resilience of random populations to the action of random perturbations [24]; universal generation of Lévy laws and $1/f$ noises [25]; universal generation of statistical self-similarity [26].

In what follows we set

$$\psi(x, y) = \int_{-\infty}^{\infty} a^2 \lambda(a, x, y)\, \mathrm{d}a \tag{2}$$

($x > 0$; $y \geqslant 0$). The function $\psi(x, y)$ is about to play a key role in what follows, and is henceforth termed the 'key function'. Note that if $\lambda(a, \omega, \tau) = \Lambda(a/\psi_*(\omega, \tau))/\psi_*(\omega, \tau)^2$—where $\Lambda(u)$ (u real) is an arbitrary non-negative valued function with a unit-valued second moment (i.e. $\int_{-\infty}^{\infty} u^2 \Lambda(u) \mathrm{d}u = 1$)—then $\psi(x, y) = \psi_*(x, y)$. Hence, any desired key function $\psi(x, y)$ can be obtained from a wide class of underlying Poissonian intensities $\lambda(a, \omega, \tau)$.

Analysis of the superposition model of equation (1) establishes an explicit transformation mapping the MSD $M_X(t) = \langle X(t)^2 \rangle$ of the impact pattern X to the variance $V_Y(t) =$

[4] Somewhat informally, the meaning of the Poissonian intensity $\lambda(a, \omega, \tau)$ is as follows: a gust with impact parameters belonging to the infinitesimal box $(a, a + \mathrm{d}a) \times (\omega, \omega + \mathrm{d}\omega) \times (\tau, \tau + \mathrm{d}\tau)$ exists with probability $\lambda(a, \omega, \tau)\, \mathrm{d}a\, \mathrm{d}\omega\, \mathrm{d}\tau$.

$\langle Y(t)^2\rangle - \langle Y(t)\rangle^2$ of the probe's trajectory Y. Indeed, using probabilistic conditioning, combined with results from the theory of Poisson processes ([18], equations (3.9)– (3.10)), yields the following connection between the aforementioned MSD and variance:

$$V_Y(t) = \int_0^\infty \int_0^1 M_X(x(1-y))\psi\left(\frac{x}{t}, ty\right) \mathrm{d}x\, \mathrm{d}y. \tag{3}$$

We term the Poissonian statistics of the impact parameters $\mathcal{P}$ '*variance-universal*' if the variance $V_Y(t)$ of the probe's trajectory Y is independent—up to a scale factor—of the impact pattern X. Namely the impact parameters' Poissonian statistics are *variance-universal* if the variance of the probe's trajectory admits the form $V_Y(t) = c_X \cdot v(t)$ where c_X is a constant depending on the impact pattern X, and where $v(t)$ is a temporal function which is independent of the impact pattern X. Equation (3), in turn, implies that *the impact parameters' Poissonian statistics are variance-universal if and only if the key function* $\psi(x, y)$ *satisfies the scaling relation*

$$\psi\left(\frac{x}{t}, ty\right) = t^\alpha \psi(x, y) \tag{4}$$

($x > 0$; $y \geqslant 0$; $t > 0$), *where* α *is an arbitrary positive exponent*. Moreover, if the scaling relation of equation (4) holds, then the probe's trajectory Y is an anomalous diffusion with exponent α:

$$V_Y(t) = V_Y(1) \cdot t^\alpha \tag{5}$$

(in other words, $c_X = V_Y(1)$ and $v(t) = t^\alpha$). In order that the variance-universal Poissonian statistics be admissible the key function $\psi(x, y)$ needs to satisfy the integrability condition $V_Y(1) < \infty$.

Variance-universal Poissonian statistics form a universal mechanism for the generation of anomalous diffusion: no matter what impact pattern X represents the gusts–probe interaction, the probe's trajectory Y can be set to display a specific anomalous diffusion behavior—with desired exponent α—by applying a proper Poissonian statistics of the impact parameters $\mathcal{P}$. The 'universality' here is in the sense that the probe's variance $V_Y(t)$ is independent—up to the scale factor $c_X = V_Y(1)$—of the choice of the impact pattern X. Interpreting the superposition model of equation (1) as a signal superposition model, we obtain that variance-universal Poissonian statistics yield anomalous diffusion output signals (Y) for whatever statistical pattern (X) of the input signals 'fed' into the communication channel.

We emphasize that the anomalous diffusion statistics of equation (5) emerged naturally—rather than were a goal we aimed at attaining. Indeed, what we sought were variance-universal Poissonian statistics—and we had no *a priori* information or prerequisites regarding the form of the temporal function $v(t)$. It so turned out that the only functional form the temporal function $v(t)$ can admit—having applied variance-universal Poissonian statistics—is that of a power-law. In turn, the power-law functional form $v(t) = t^\alpha$ is the hallmark of anomalous diffusion.

Two related works are [25] and [26]. In [25] a superposition model analogous to equation (1)—yielding stationary trajectories Y—was considered. The Poissonian statistics of the impact parameters was termed: (i) 'amplitude-universal' if the stationary distribution of the trajectory Y is independent—up to a scale factor—of the impact pattern X; (ii) 'temporal-universal' if the power spectrum of the trajectory Y is independent—up to a scale factor—of the impact pattern X. The classes of amplitude-universal and temporal-universal Poissonian statistics were characterized, and their corresponding stationary laws and power spectra were shown to coincide, respectively, with the classes of Lévy laws and $1/f$ noises—thus providing a universal explanation to the ubiquity of these 'fractal statistics' (in the context of stationary processes).

In [26] the superposition model of equation (1) was considered, and statistical self-similarity [27] was sought: Poissonian statistics of the impact parameters that render the trajectory Y statistically self-similar—the self-similarity being independent of the choice of the impact pattern X. Finite-variance statistically self-similar random trajectories display both $1/f$ noise behavior, and anomalous diffusion behavior. Hence, the mechanism introduced in [26]—in the case of finite variance trajectories—further produces anomalous diffusion. Yet, it produces anomalous diffusion via statistical self-similarity—and is thus much more restrictive than the mechanism presented in this communication. Indeed, the class of 'variance-universal' Poissonian statistics is far larger than the class of Poissonian statistics yielding statistical self-similarity.

The $1/f$ noises obtained in [25] via the notion of 'temporal universality' yield power spectra of the form $1/|f|^\beta$, with exponent β taking values in the range $0 < \beta < 1$. On the other hand, the $1/f$ noises obtained in [26] via statistical self-similarity yield power spectra of the form $1/|f|^\beta$, with exponent β taking values in the range $\beta > 1$. These two $1/f$ noise ranges are the 'spectral analogs' of the two anomalous diffusion ranges ($0 < \alpha < 1$ and $\alpha > 1$). Establishing a general $1/f$ noise theory which is parallel to the anomalous diffusion theory presented in this communication—i.e. a $1/f$ noise theory based on the superposition model of equation (1), and spanning both exponent ranges $0 < \beta < 1$ and $\beta > 1$—is a goal set for further research.

A fairly general class of key functions $\psi(x, y)$ satisfying the scaling relation of equation (4) is given by

$$\psi(x, y) = \phi(xy)y^\alpha \tag{6}$$

($x > 0$; $y \geqslant 0$), where $\phi(u)$ ($u \geqslant 0$) is an arbitrary non-negative valued function for which the integrability condition $V_Y(1) < \infty$ holds. Examples of the generation of anomalous diffusion via the key function of equation (6) are presented in table 1. Examples 1–4 demonstrate how anomalous diffusion can emerge from the superposition of mundanely 'regular' impact patterns: (1) stationary patterns; (2) diffusion patterns (random walks, Brownian motion, finite-variance Lévy motions); (3) geometric patterns (multiplicative random walks, geometric Brownian motion, geometric Lévy motions); (4) Ornstein–Uhlenbeck patterns (generated by Langevin dynamics [28])[5]. Example (5) considers anomalous diffusion impact patterns, and generalizes examples (1) and (2) (corresponding, respectively, to the special cases $\beta = 0$ and $\beta = 1$).

Two special scenarios of the superposition model of equation (1) are the 'big bang scenario' and the 'steady state scenario'. In the big bang scenario all gusts initiate at time 0 (i.e. $\tau_k \equiv 0$), and hence the Poissonian intensity is given by $\lambda(a, \omega, \tau) = \tilde{\lambda}(a, \omega)\delta(\tau)$ ($\delta(\cdot)$ denoting the Dirac 'delta function'). In the steady state scenario gusts with amplitude a and frequency ω initiate—randomly in time—at rate $\tilde{\lambda}(a, \omega)$, and hence the Poissonian intensity is given by $\lambda(a, \omega, \tau) = \tilde{\lambda}(a, \omega)$. For both these scenarios we set

$$\tilde{\psi}(\omega) = \int_{-\infty}^{\infty} a^2 \tilde{\lambda}(a, \omega)\, \mathrm{d}a \tag{7}$$

($\omega > 0$). The function $\tilde{\psi}(\omega)$ will now assume the role of the key function $\psi(x, y)$. As in the case of the key function $\psi(x, y)$, note that if $\tilde{\lambda}(a, \omega) = \Lambda(a/\tilde{\psi}_*(\omega))/\tilde{\psi}_*(\omega)^2$—where $\Lambda(u)$ (u real) is an arbitrary non-negative valued function with a unit-valued second moment (i.e. $\int_{-\infty}^{\infty} u^2\Lambda(u)\mathrm{d}u = 1$)—then $\tilde{\psi}(\omega) = \tilde{\psi}_*(\omega)$. Hence, any desired function $\tilde{\psi}(\omega)$ can be obtained from a wide class of underlying Poissonian intensities $\tilde{\lambda}(a, \omega)$.

[5] Note that in this example the impact patterns' short-term behavior is diffusive ($M_X(t) \approx c\kappa t$ for $t \ll 1$), whereas the long-term behavior is stationary ($M_X(t) \approx c$ for $t \gg 1$).

Table 1. Generating anomalous diffusion via the key function of equation (6). The first and second columns specify, respectively, the stochastic dynamics and the MSD of the impact pattern X considered (the coefficient c, and the exponents κ and β, are arbitrary positive parameters). The third column specifies the finite-moment condition the function $\phi(u)$ needs to satisfy so as to assure that the integrability condition $V_Y(1) < \infty$ holds. The fourth column specifies the range of admissible exponents α.

Dynamics	MSD $M_X(t) =$	$\int_0^\infty u^m \phi(u)\,\mathrm{d}u < \infty$	Exponent
1. Stationary	c	$m = 0$	$\alpha > 0$
2. Diffusion	ct	$m = 1$	$\alpha > 1$
3. Geometric	$c\exp(-\kappa t)$	$m = 0$	$\alpha > 0$
4. Langevin	$c(1 - \exp(-\kappa t))$	$m = 0$	$\alpha > 0$
5. Anomalous diffusion	ct^β	$m = \beta$	$\alpha > \beta$

The counterparts of equation (3) are (i) big bang scenario:

$$V_Y(t) = \frac{1}{t}\int_0^\infty M_X(x)\tilde{\psi}\left(\frac{x}{t}\right)\mathrm{d}x \tag{8}$$

($t > 0$); (ii) steady state scenario:

$$V_Y(t) = \int_0^\infty \left(\frac{1}{x}\int_0^x M_X(y)\mathrm{d}y\right)\tilde{\psi}\left(\frac{x}{t}\right)\mathrm{d}x \tag{9}$$

($t > 0$).

Equations (8) and (9) imply that in both the big bang scenario and the steady state scenario the Poissonian statistics are variance-universal if and only if the function $\tilde{\psi}(\omega)$ is homogeneous—in which case the probe's trajectory Y is an anomalous diffusion. More precisely, (i) in the big bang scenario the probe's trajectory is an anomalous diffusion with exponent α if and only if the function $\tilde{\psi}(\omega)$ is homogeneous of order $-\alpha - 1$, and the integrability condition $\int_0^\infty M_X(u)u^{-\alpha-1}\mathrm{d}u < \infty$ is satisfied; (ii) in the steady state scenario the probe's trajectory is an anomalous diffusion with exponent α if and only if the function $\tilde{\psi}(\omega)$ is homogeneous of order $-\alpha$, and the integrability condition $\int_0^\infty M_X(u)u^{-\alpha}\mathrm{d}u < \infty$ is satisfied.

In both the big bang scenario and the steady state scenario the following examples of table 1 are non-admissible impact patterns: stationary patterns (example 1); diffusion patterns (example 2); anomalous diffusion patterns (example 5). Geometric patterns (example 3) are admissible impact patterns in the big bang scenario alone—yielding subdiffusive trajectories ($0 < \alpha < 1$). Ornstein–Uhlenbeck patterns (example 4) are admissible impact patterns in both scenarios—yielding subdiffusive trajectories ($0 < \alpha < 1$) in the big bang scenario, and yielding superdiffusive trajectories (with exponent $1 < \alpha < 2$) in the steady state scenario.

In this communication we considered the superposition model of equation (1), which describes the random motion of a probe tossed into a stochastic bath. The probe's trajectory Y is the superposition of the effects of all gusts impacting it. The gusts are i.i.d. and share a statistically common impact pattern X representing the gusts–probe interaction, and each gust has its own impact parameters—amplitude, frequency and initiation epoch. Our aim was to characterize the stochastic-bath statistics—conveyed by the Poissonian statistics of the impact parameters $\mathcal{P}$—which are variance-universal: rendering the variance of the trajectory Y independent, up to a scale factor, of the impact pattern X.

Analysis showed that the variance-universal Poissonian statistics are characterized by the scaling relation of equation (4), and that the only possible corresponding probe trajectories are anomalous diffusions. Namely given a desired exponent α—appropriate variance-universal

Poissonian statistics will render the probe's trajectory Y an anomalous diffusion with exponent α, for whatever impact pattern X. We have thus established a novel universal mechanism for the generation of anomalous diffusion, which is fundamentally different of the commonly applied Lévy and fBm anomalous diffusion models: Lévy halts and anti-persistent fBm generating subdiffusion, and Lévy walks and persistent fBm generating superdiffusion.

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